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Another look at the spheres

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Abstract

In this work, I provide another method of finding the surface area of a sphere, by projecting each point on the sphere to the faces of a cube. To understand how I do this, first, I demonstrate a similar method but in two dimensions by breaking up the circumference of a circle and projecting it to sides of a square. In each subsection, I start with defining these projections, before moving on to finding the circumference/surface area.

Key words and phrases: Sphere, Integration, Projection

Otra mirada a la esfera

Resumen

En este trabajo, proporciono otro método para encontrar el área de la superficie de una esfera, proyectando cada punto de la esfera a las caras de un cubo. Para entender cómo hago esto, primero, demuestro un método similar pero en dos dimensiones al romper la circunferencia de un círculo y proyectarlo a los lados de un cuadrado. En cada subsección, comienzo definiendo estas proyecciones antes de pasar a buscar la circunferencia/ área de superficie.

Palabras y frases clave: Esfera, Integración, Proyección

1 Introduction

When we look at a sphere from a given direction, it looks just like a circle with area πr^2 where r is the radius of the sphere. When I first found out that the surface area of a sphere is $4\pi r^2$, I wanted to find an intuitive reason why the surface area is four times the area of the circle which we see when we look at the sphere from one direction. In order to do so, I wanted to break up the surface of the sphere and reconstruct four circles of radius r from it. Although this is not what I ended up doing, the main idea in this work of projecting a sphere to a flat surface was motivated in this way.

2000 years ago, Archimedes also used projection in order to find the surface area of the sphere. In Archimedes' Hat Box Theorem he stated that the surface area of a sphere is conserved when it is projected to the label of a cylinder exactly containing the sphere. This means that the surface area of a sphere equals the surface area of the label of the cylinder with hight $2r$, and circumference $2\pi r$ hence proving that the surface area of the label equals the surface area of the sphere $= 2\pi r \times 2r = 4\pi r^2$. [1].

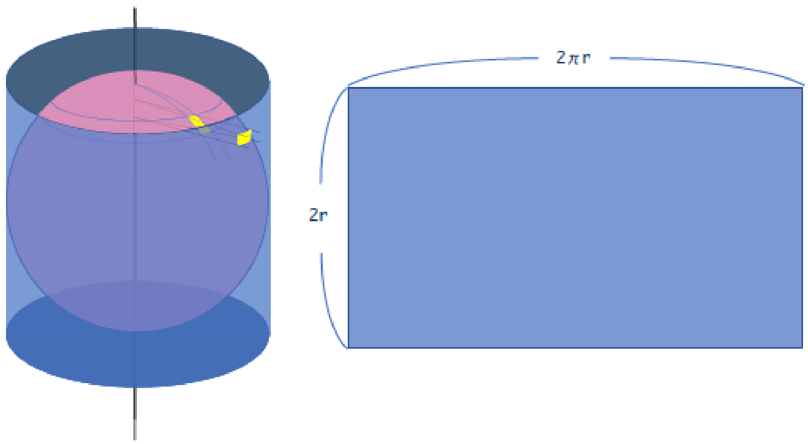

Figure 1:

The main difference between our proof and Archimedes' is that in our proof, we project the sphere to the faces of a cube instead of a cylinder. By introducing the idea of "point densities", we project one point on the sphere to multiple surfaces simultaneously, enabling the sphere to be "broken up" to the different faces of the cube.

2 Defining densities and projections

Firstly, we introduce the terms and techniques which we use in order to obtain circumference of the circle.

2.1 Point density and Solid Line Value

For every point on a line, let us assign a density to that point, where 1 is the default density. For example, a point of density 0.5 can be thought of as a point that is half transparent, like a point drawn by a highlighter.

Let us denote $D(x)$ the density of the point x.
Then let us define the solid line value of L, $S(L)$, as

$$S(L) = \int_0^l D(x)\,\mathrm{d}x$$

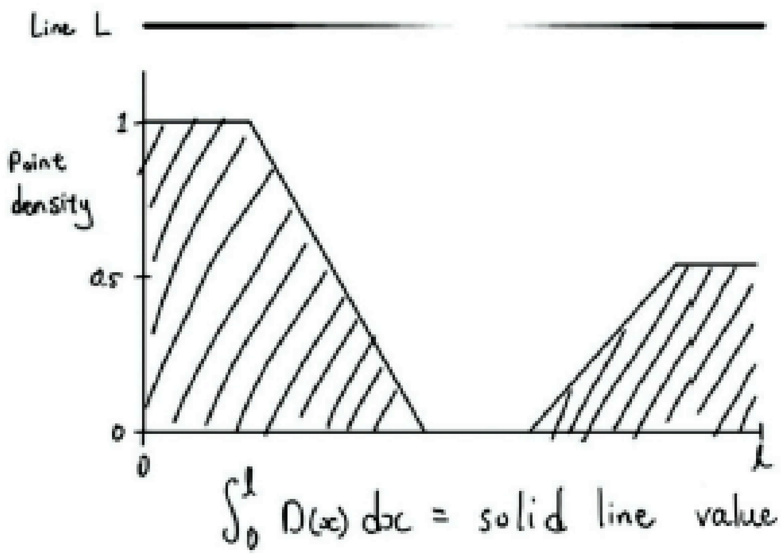

Figure 2:

2.2 Projecting a line at an angle

Consider the line L of length l, with uniform density d, which makes an angle θ with the x axis. We define a projection of this onto the x axis where line L is mapped to its shadow L' in such a way so that the solid line value is conserved (i.e. $S(L') = S(L)$). As the length of $S(L')$ is $l\cos\theta$, we define L' to have uniform density $\frac{d}{\cos\theta}$ so that:

$$S(L') = \frac{d}{\cos\theta} \times l\cos\theta = d \times l = S(L)$$

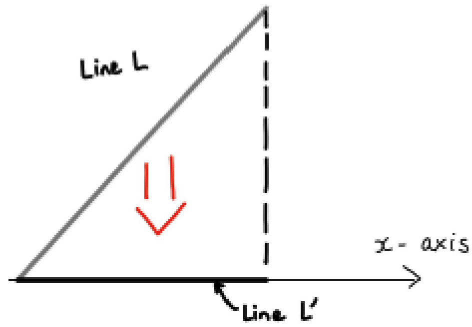

Figure 3:

2.3 Orthogonal projections

We define an orthogonal projection as a decomposition and projection of a line of uniform density 1 onto both the x and y axes.

Firstly, we separate line L into two lines, L_1 and L_2 so that they have uniform density $\cos^2\theta$ and $\sin^2\theta$ respectively.

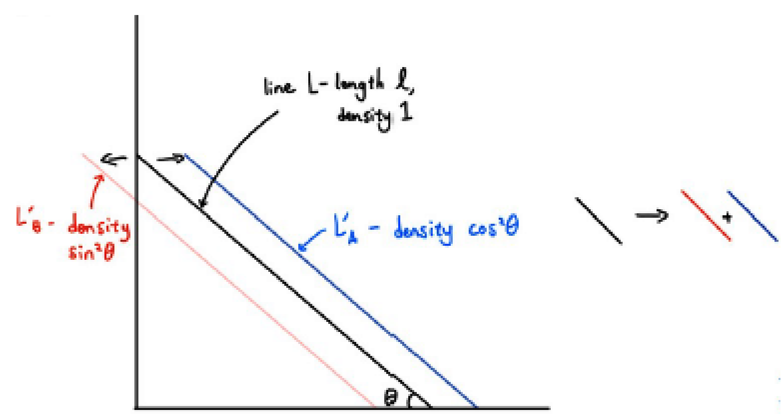

Figure 4:

$$S(L_1) + S(L_2) = (\cos^2\theta + \sin^2\theta) \times S(L) = S(L).$$

We project the two lines L_1 and L_2 onto the x axis and y axis creating two images, L_A and L_B respectively.

By the definition of a projection, $S(L_A) = S(L_1)$ and

$$\text{density* of } L_A = \text{density of } L_1 \times \frac{1}{\cos\theta} = \cos^2\theta \times \frac{1}{\cos\theta} = \cos\theta$$

Similarly $S(L_B) = S(L_2)$ and density of $L_B = \sin\theta$.

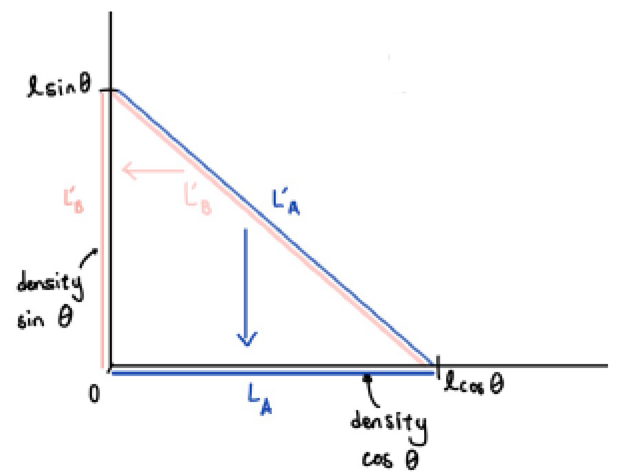

Figure 5:

Hence

$$S(L_A) + S(L_B) = S(L_1) + S(L_2) = S(L).$$

So for any angle θ, an orthogonal projection can decompose line L into two lines, L_A and L_B with density $\cos\theta$ and $\sin\theta$.

(Here, we define the density of a line to be d when the line has uniform point density of d throughout.)

3 Obtaining the circumference of a circle

In a similar way to the orthogonal projection of a line, we can project a circle onto sides of a square.

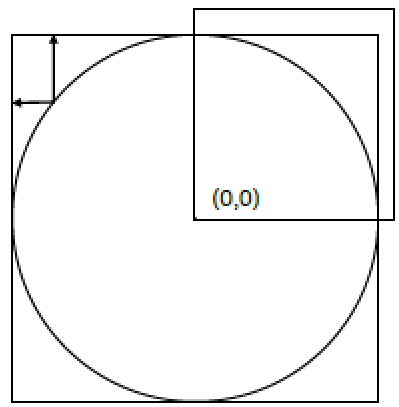

Figure 6:

Due to the symmetry of the projection, each quarter of a circle is decomposed in exactly the same way. Therefore, it suffices to show what happens to the top right corner only.

A circle can be interpreted as the limit of a regular polygon as the number sides tends to infinity. For a regular polygon, we can apply the orthogonal projections. By looking at the density distribution on the x axis as the number of sides tends to ∞, we will determine the density distribution obtained by applying the orthogonal projection on the circle.

We create a quarter of a polygon by dividing the quarter circle into n equal arcs and connecting its end points with straight lines.

Let $n\theta = \frac{\pi}{2}$. Let R be the radius of the circle. P_0 to P_n are endpoints of the arc where P_i has coordinates $(R\cos i\theta, R\sin i\theta)$, and L_1 to L_n are the segments joining them. The gradient of line $L_i = P_iP_{i-1}$ is greater than the gradient of the tangent to the circle at P_{i-1} and less than the gradient of the tangent to the circle at P_i. As the tangent at P_i is perpendicular to $\overline{OP_i}$, the tangent at P_i makes an angle $\frac{\pi}{2} - i\theta$ with the x axis. So for each angle α_i that L_i makes with the x axis,

$$\frac{\pi}{2} - i\theta < \alpha_i < \frac{\pi}{2} - (i-1)\theta.$$

Let L_A and L_B be the results of the projection of the quarter polygon onto the x axis and y axis respectively. So for all points x on L_A, $\exists i : 1 \leq i \leq n$

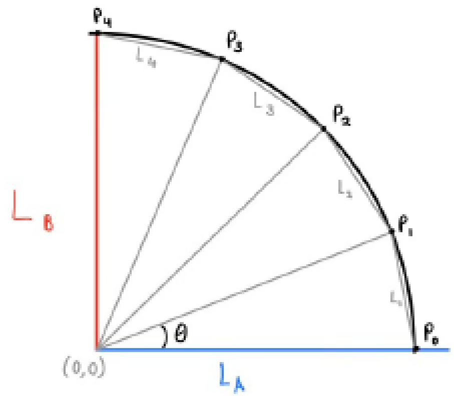

Figure 7:

so that

$$R\cos i\theta \le x < R\cos(i-1)\theta \Rightarrow D(x) = \cos\alpha_i \Rightarrow \sin(i-1)\theta < D(x) < \sin i\theta.$$

Let $\gamma : 0 \le \gamma < \frac{\pi}{2}$ be such that, $x = R\cos\gamma$. Since $\exists i : (i-1)\theta < \gamma \le i\theta$

$$\sin(\gamma-\theta) \le \sin(i-1)\theta < D(x) \le \sin i\theta < \sin(\gamma+\theta)$$

As the polygon tends towards a circle, $n \to \infty$ and $\theta \to 0$.

$$\sin(\gamma-\theta) \to \sin\gamma \text{ and } \sin(\gamma+\theta) \to \sin\gamma$$

So for a circle,

$$D(x) = D(R\cos\gamma) = \sin\gamma$$

Because the way in which densities are assigned to the x and y axes is symmetrical, $S(L_A) = S(L_B)$. As the overall line value is conserved in projection, $S(L_A) + S(L_B) = 2S(L_A) = S(\frac{1}{4}\text{circumference})$. So

$$4 \times 2S(L_A) = 8S(L_A) = \text{ circumference of whole circle}$$

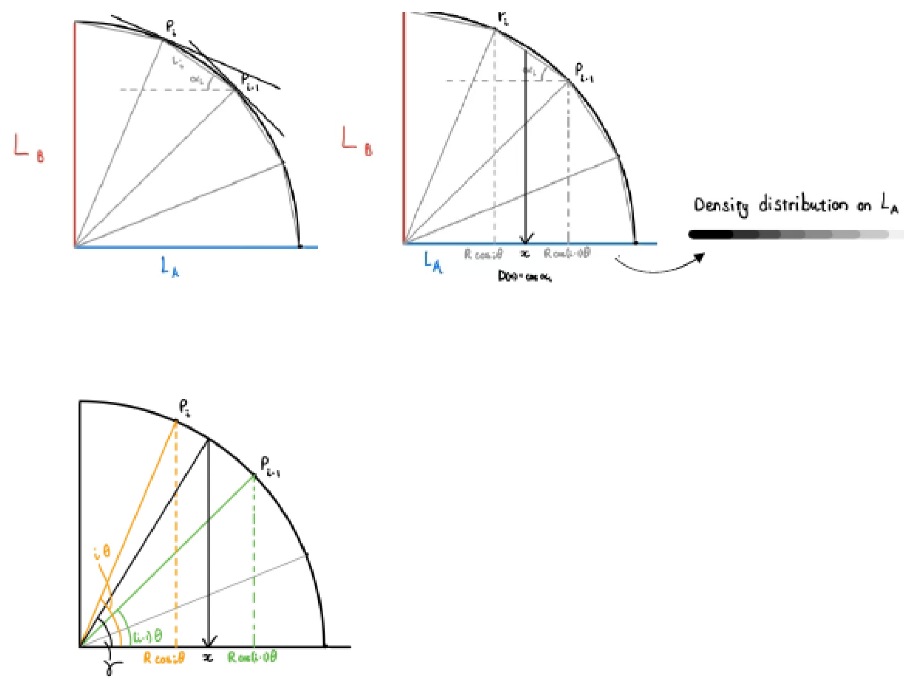

Figure 8:

$$
\begin{aligned}
S(L_A) &= \int_0^R D(x)\,\mathrm{d}x = \int_0^R D(R\cos\gamma)\,\mathrm{d}(R\cos\gamma) \\
&= \int_{\frac{\pi}{2}}^0 \sin\gamma(-R\sin\gamma)\,\mathrm{d}\gamma \\
&= R\int_{\frac{\pi}{2}}^0 -\sin^2\gamma\,\mathrm{d}\gamma = R\int_0^{\frac{\pi}{2}} \sin^2\gamma\,\mathrm{d}\gamma \\
&= R\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\gamma}{2}\,\mathrm{d}\gamma \\
&= R\Big[\frac{\gamma}{2} - \frac{\sin 2\gamma}{4}\Big]_0^{\frac{\pi}{2}} \\
&= \frac{R\pi}{4}
\end{aligned}
$$

$$\Rightarrow \text{circumference} = 8S(L_A) = 8 \times \frac{R\pi}{4} = 2\pi R.$$

4 Application in 3 dimensions

4.1 Solid Area Value

We define the solid area value $S(P)$ of an area P in a similar way to the solid length value:

$$\begin{aligned} S(P) &= \iint_P D(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_P D(r,\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned}$$

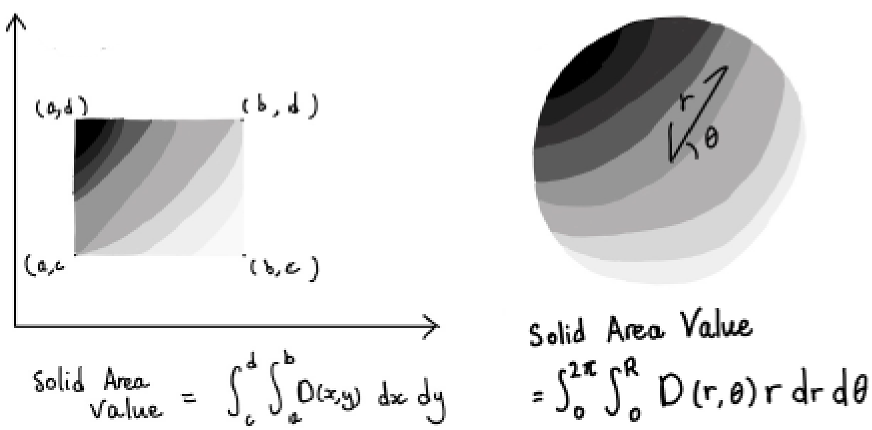

Figure 9:

4.2 Projecting a plane at an angle

Consider the shape P, with uniform density d which is in a plane which makes an angle θ with the xy plane. We define a projection of this onto the xy plane where P is projected to its shadow P' in such a way so that solid area value is conserved (i.e. $S(P') = S(P)$). The width of P' and P are equal, and the height of P' is $h\cos\theta$, hence

$$\text{area of } P' = \text{area of } P \times \cos\theta.$$

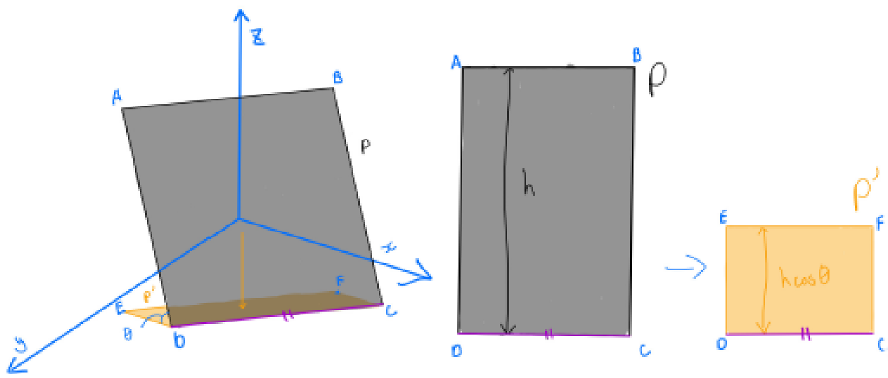

Figure 10:

We define P' to have a uniform density of $\frac{d}{\cos\theta}$ throughout, so that:

$$S(P') = \frac{d}{\cos\theta}\times\text{area of } P' = \frac{d}{\cos\theta}\times\cos\theta\times\text{area of } P = d\times\text{area of } P = S(P).$$

4.3 Decomposing a shape in three dimensions

Let α, β, γ be the angles the plane Q containing the shape P makes with the yz, zx, and xy planes respectively.

Firstly, we prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Let $\boldsymbol{n}$ be the vector normal to Q through the origin, meeting Q at (a, b, c). Then

$$\begin{aligned} a &= |n|\cos\alpha \\ b &= |n|\cos\beta \\ c &= |n|\cos\gamma \end{aligned}$$

$$\begin{aligned} \Rightarrow |n|^2 &= a^2 + b^2 + c^2 \\ &= |n|^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \end{aligned}$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

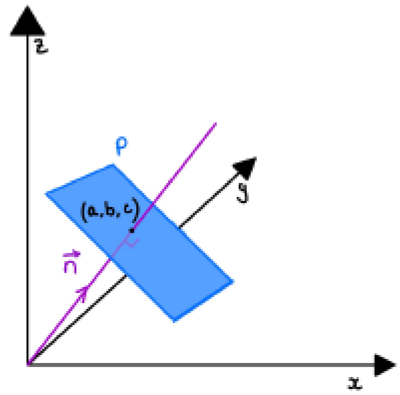

Figure 11:

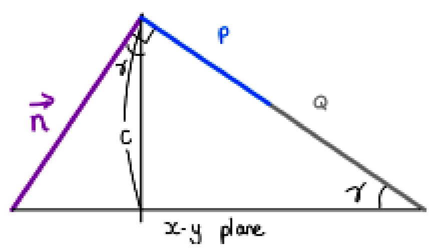

Figure 12:

4.4 Orthogonal projections in three dimensions

Let P be a shape with uniform density 1. And let $\alpha,\ \beta,\ \gamma$ be the angles shape P makes with the $yz,\ zx$ and xy planes respectively. In a similar way as the orthogonal projections in two dimensions, we separate P into three shapes P_1, P_2 and P_3 with uniform density $\cos^2\alpha, \cos^2\beta$ and $\cos^2\gamma$ respectively.

$$S(P_1)+S(P_2)+S(P_3) = (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)\times S(P) = S(P).$$

Figures 13 and 14 are special cases of this projection where the original shape is a triangle. However, this method works for any shape, as the ratio of the solid area value of each of the projected shape to the original shape is only dependent on the angle of the projections.

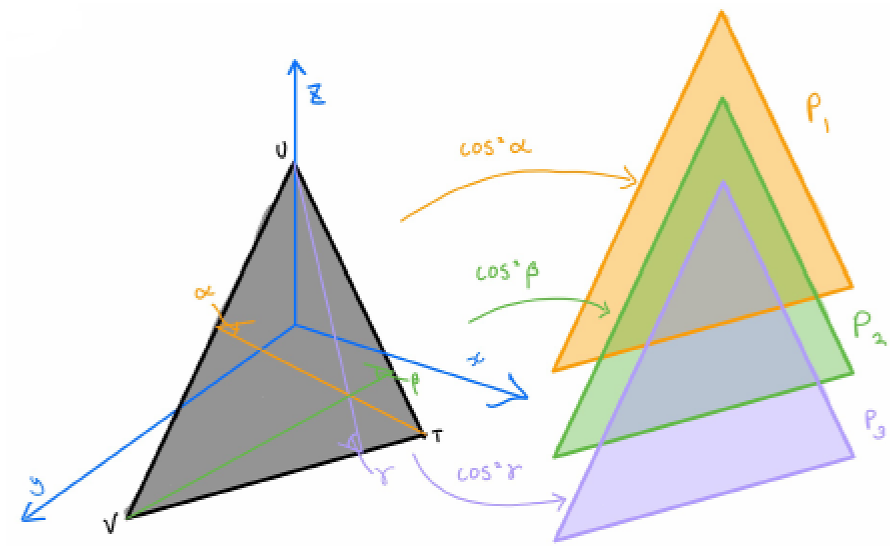

Figure 13:

We then project P_1, P_2 and P_3 perpendicularly to the yz, zx and xy planes to create images A, B and C, respectively.

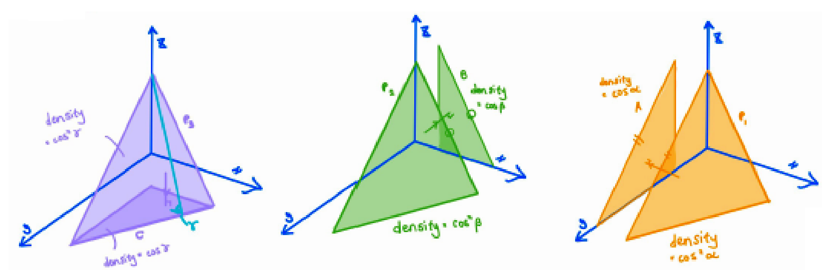

Figure 14:

By the definition of a projection, $S(P_1) = S(A)$ and

$$\text{density** of } A = \text{density of } P_1 \times \frac{1}{\cos\alpha} = \cos^2\alpha \times \frac{1}{\cos\alpha} = \cos\alpha.$$

Similarly $S(P_2) = S(B)$, density of $B = \cos\beta$, $S(P_3) = S(C)$ and density of $C = \cos\gamma$.

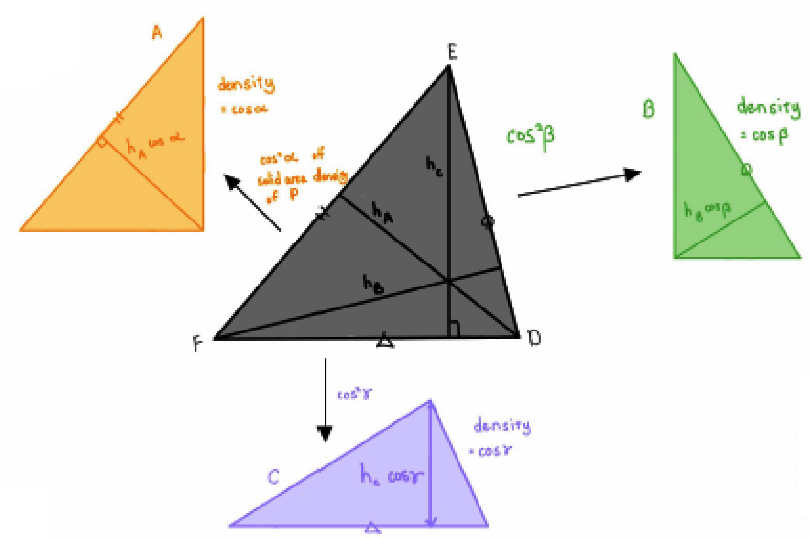

Figure 15:

Hence

$$S(A) + S(B) + S(C) = S(P_1) + S(P_2) + S(P_3) = S(P)$$

So the orthogonal projection decomposes a shape of density 1 to three shapes with density $\cos\alpha, \cos\beta$ and $\cos\gamma$.

(Here we define the density of a plane to be d, when the plane has uniform point density of d throughout.)

5 Obtaining the surface area of a sphere

5.1 Projecting a sphere to a cube

For the surface area of the sphere with equation $x^2+y^2+z^2 = R^2$, we project the vertices of the polyhedron onto the six faces of a cube with equations,

$$x = \pm R \quad y = \pm R \quad z = \pm R.$$

As points will be projected to each of the six faces in the same way, we will calculate the solid area value of the plane $z = -R$ only. Each point will

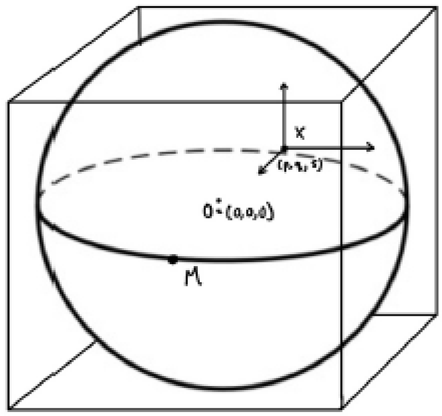

Figure 16:

be projected on to one xy, one yz and one zx plane depending on which of the two parallel planes the point is closer to.

As with the circle, we can approximate the sphere by a polyhedron taking points on the sphere at regular intervals and connecting them to makes faces.

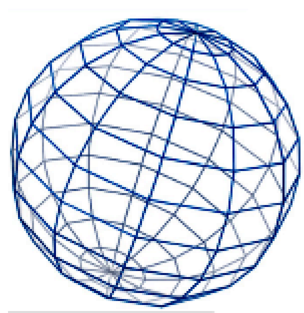

Figure 17:

The sphere can be interpreted as the limit as the number of faces tends towards infinity. Therefore, for a point on the sphere with coordinates p, q and s, we perform the orthogonal projection of that point treating it as an infinitely small plane with inclination equivalent to that of the tangent at that point.

To work out the angle of intersection of the tangent at V with $z = -R$, we take the plane containing $M = (0,\, 0,\, -R)$, V, and O.

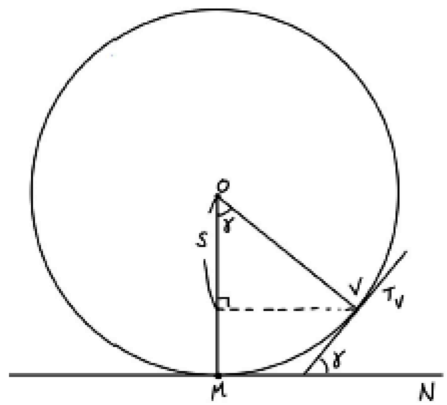

Figure 18:

OM is perpendicular to the xy plane (call this N), and OV is perpendicular to the tangent plane at V (call this T_V), so the angle at which lines T_V and N meet in this cross section is the angle of intersection of the planes T_V and N. Let $\gamma = \angle VOM$. As OM is perpendicular to the xy plane, dropping a perpendicular onto OM from V preserves the z coordinate, so $|OV| \cos\gamma = R\cos\gamma = |s|$. By the definition of a projection, we know that the density projected onto point $(p,\, q)$ on N is equal to cos of the angle between N and the tangent plane at $(p,\, q,\, s)$. Therefore:

$$\text{density at point } (p,\, q) = \cos\gamma = \frac{|s|}{R} = \frac{\sqrt{R^2 - p^2 - q^2}}{R}.$$

5.2 Calculating the solid area value of each face

To calculate the solid area value of the circle with varying point density, we use polar coordinates. We integrate as r varies between 0 and R(the radius of the projected circle), and as θ varies between 0 and 2π.
Let $r = \sqrt{p^2 + q^2}$ then

$$D(r,\, \theta) = \frac{\sqrt{R^2 - r^2}}{R}$$

$$
\begin{aligned}
S(N) &= \int_0^{2\pi} \int_0^R D(r,\, \theta) r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^R \frac{\sqrt{R^2 - r^2}}{R} r \,\mathrm{d}r \,\mathrm{d}\theta \\
\text{Let } r = R\sin\sigma \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\sigma}\, R\sin\sigma \,\mathrm{d}R\sin\sigma \,\mathrm{d}\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\sigma R\sin\sigma \frac{\mathrm{d}R\sin\sigma}{\mathrm{d}\sigma} \,\mathrm{d}\sigma \,\mathrm{d}\theta \\
&= R^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^2\sigma \sin\sigma \,\mathrm{d}\sigma \,\mathrm{d}\theta \\
&= R^2 \int_0^{2\pi} \left[-\frac{\cos^3\sigma}{3} \right]_0^{\frac{\pi}{2}} \mathrm{d}\theta \\
&= R^2 \int_0^{2\pi} \frac{1}{3} \,\mathrm{d}\theta \\
&= R^2 \frac{2\pi}{3}
\end{aligned}
$$

As this is one of six faces, overall, the surface area of the sphere is

$$
R^2 \frac{2\pi}{3} \times 6 = 4\pi R^2
$$

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6 Summary

In this work, I introduced a different method of obtaining the surface area of a sphere. By introducing the idea of points with "densities", I reduced a three dimensional problem to an integration problem in two dimensions. The method of projecting and decomposing a curved surface can be applied to any curved surface that is differentiable. Therefore it may be possible to use this method to find the surface area of other curved surfaces.

7 Acknowledgements

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References

[1] Donald Simanek, Lock Haven University, *Archimedes' Hat-Box Theorem*, https://www.lockhaven.edu/~dsimanek/puzzles/hatbox.html

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