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The Cultural Impact of Hindu-Arabic Numbers[1]

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To my wife, Carely, and my son, Tomasito, with love.

Abstract

Hindu-Arabic numerals are the result of a gradual evolution of positional notation. The main evolutionary line started with the Brahami species in India around the V century, and then, it evolved to the Gvalior species. Then we found the Hindi/Gubar species in Persia and Arabia, after the VIII and IX centuries. Finally, the notation reached medieval Europe with the Apex species and then the Algorisms species, both of which consolidated during the XVI and XVII Renaissance centuries. Since then, there are no records of any evolutionary changes of the current species.

This paper proposes that the evolution of the notation was through the graphemic alteration over each historical period. This makes Hindu-Arabic notation the great legacy that both civilizations contributed to humankind.

Key words and phrases: Symbolic number systems, Roman numbers, Placed-Valued number systems, Hindu-Arabic numbers, Evolution of numbers.

El Impacto Cultural de los Números Indoarábigos

Resumen

Los númerales indo-arábigos son el producto de una evolución gradual de una notación posicional. La principal línea evolutiva comienzó con la especie brahami en la India alrededor del siglo V, y después evolucionó en la especie Gvailor. Luego encontramos la especie hindi/gubar en Persia y Arabia, durante los siglos VIII y XI. Finalmente,

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la notación llegó a la Europa medieval con las especies ápices y posteriormente la especie algorismos, que se consolida durante los siglos XVI y XVII del renacimiento. Desde entonces, no se registra ningún cambio evolutivo de la especie actual.

En este artículo propone que la evolución de la notación se produjo por medio de la alteración grafémica en cada período histórico. Esto hace de la notación indo-arábiga el gran aporte que ambas civilizaciones dejaron a la humanidad.

Palabras y frases clave: Sistemas de numeración simbólicos, números romanos, sistemas de numeración posicionales, números indoarábigos, evolución de los números.

1 Introduction

The numerals 1,2,3,4,5,6,7,8,9 and 0 that we use currently result from an evolutionary change of species, which mainly started with Brahami numbers in India and evolved to Gvalior numbers. The Hindi/Gubar numbers appeared in the works of al-Khwārismi during the Persian Empire until Gerbert D'Aurillac brought the notation to Europe, and the Apex numbers appeared. Finally, Leonardo Fibonacci was, among others, responsible for introducing Algorisms numbers. Thank to these people, convinced of the great advantage of the positional system, Hindu-Arabic notation consolidated over Roman Notation.

However, the assimilation of the Hindu-Arabic notation met resistance when moving from one civilization to another. For many reasons, the notation met a strong intercultural opposition whether religious motives (See Menninger [3], page 400), prejudice againdt of the unknown, or merely due to conservativism in the receiving culture (See Ifrah [2], pages 539-541).

But beyond the motivations that each civilization had to reject the new notation, what determined the triumph of Hindu-Arabic notation against Roman notation was the idea to move from a symbolic system, where the numeral had an absolute value, to a positional system where the numeral had a relative value. Hindu-Arabic notation allowed the representation of high quantities with ten numerals – from 1 to 9, including 0 as the digit's position in the figure shows a change in the magnitude's order of the number. Since Roman was symbolic it difficulty representing high quantities, as it only had the numerals I, V, X, L, C, D, and M vinculum was used to multipliy by ten the absolute value of the numeral (e.g., twenty-five thousand forty-three is written as to represent large figures $\overline{\text{XXV}}$DCCXLIII)) . The result was cumbersome. It was confusing, especially with the numbers that allowed for more than one representation (e.g., four could be IIII as in on clocks or IV in standard form).

On the other hand, the representation of a figure in Hindu-Arabic numerals was unique. The digits in any number under this notation were the scalars of a linear combination of powers of ten (e.g., $45781 = 4 \times 10^4 + 5 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 1 \times 10^0$). Another advantage of Hindu-Arabic notation was the simplification of arithmetical calculations. The time invested in performing elementary operations was significantly less than with Roman notation.

We can find many more other reasons that explain the current usage of Hindu-Arabic notation, practically unchanged, from the Renaissance to now. Except artistic or creative modification, there have not been motivations that justify the evolution of the current notation to a new species: on the contrary, technological innovations have standardized even more Hindu-Arabic notation worldwide. Only the *Hindi* numeral species (Eastern Arabic numerals) are currently used in the majority of the countries in the Near East (see Ifrah [2], page 368), and in Greece and Israel, where the letters of the Greek and Hebrew alphabet, as numerals (see Cajori [1], page 19 and 25).

This paper proposes graphemic alteration as an evolutive cause of Hindu-Arabic notation through incorporation and removal of lines in the numerals'. To that end, we have structured the work as follows: In the first section, we make a brief comparison between placed-valued systems and symbolic systems. The second section is devoted to zero and ciphering considered at to the idea of placed-valued notation. In the third section, we describe the journey of Hindu-Arabic notation from India to Medieval Europe, previously passing through Persia and Arabia, highlighting the works of al-Khwārizmi, Gerbert D'Aurillac, and Leonardo Fibonacci as main contributers to this theory. In the next section, we show each numeral's evolutive branches (established by Ifrah [2]), we propose the sequence Brahami-Gvalior-Hindi/Gubar-Apex-Algorisms as the main evolutionary branch of Hindu-Arabic notation, and we propose this work's central idea: the graphemic alteration through addition or subtraction of the numbers sketch. The paper concludes with a final reflection where we postulate that Hindu-Arabic numerals are the main legacy that Indian civilization and the Arabic civilization left to humankind.

Throughout this article we use the word number to refer to its double meaning as number and numeral.

2 Place-Valued Systems and Symbolic Systems

To understand why it is Hindu-Arabic notation and not Roman notation that we use in our day-to-day life, we have to establish the difference between a placed-valued system and a symbolic system. A **symbolic** number system is a set of graphemes in which each has an absolute value. In the Roman system, I represents the one, V represents five units, X represents ten, C represents hundred, D represents five hundred, and M represents thousand. Regardless where one places the grapheme, the number value remains unaltered. The main problem that symbolic systems face is the ambiguity in the quantitiy representation. We do not know if, during Roman time, there were standardized rules about the writing of numbers; empirical evidence indicates that possibly there were no such rules. The next illustration [2] shows a clock where we can see the four as IIII, in place of the standard notation as IV.

Figure 1: Clock with Roman Numbers.

On the other hand, a **placed-valued** number system is a set of graphemes where each one has a relative value. Unlike symbolic systems, inside placed-valued systems, the grapheme's number value depends on its placement within the number one wants to represent. In the case of Hindu-Arabic notation, each grapheme increases its value into power of ten running as the number moves from right to left. Therefore, any number can be represented with the symbols 1 to 9. But, do we represent the tens, the hundreds, the thousands, and higher orders? A symbol is required torepresent the absence

[2] Placed in at the Foley Library of Gonzaga University in Spokane, WA. Photo from the author.

of a quantity in each magnitude order, that is, a grapheme to represent the nothingness. In other words, the success of place-valued systems was *zero*. We discuss this fantastic idea in the next section.

3 The zero

In Menninger's book *Number Words and Number Symbols. A Cultural History of Numbers* ([3], page 396) we find:

"The springboard for the positional principle was not ordering and grouping but "encipherment," i.e., the assigning of an individual digit to each of the first nine numbers".

Menninger's claim invites us to reflect on what is behind placed-valued systems. Hindu-Arabic notation is so familiar to us that sometimes we do not realize what Menninger wants to suggets is that the encipherment of the first nine first numbers represents a total rupture with Roman notation. To represent a number in Roman notation, according to modern standards, symbols must be gropued in groups of three, with greater values placed on the left lesser values on the right. The only exception is placing a symbol of less value to the left, which can only occur if the two symbols are of the same magnitude order (e.g., the writing of forty-nine is XLXI instead of IL). In the first section, we discussed that the modern rules of Roman notation probably did not have exist in this historical period. This would generate ambiguities in such numbers with several representations. Another characteristic of the Roman system is that the symbols change by multiples of five. Therefore conclude that Roman numbers are likely a quinary –of base 5–. However, this is not the Roman system's weakness, as a number system can have any number as the base. The Roman system's real issue is that its notational ambiguity prevents the uniqueness of the representation of a number in terms of a linear combination. In other words, Roman numbers do not represent the coordinates of a quinary or decimal base for Natural numbers, Hindu-Arabic numbers do, hence, their strength.

To represent a Natural number, it was not only necessary, as Menninger claims, to give an individual value to each number from one to nine. It was also neccesary to invent a notation that could represent the change in the magnitude order. In other words, it was neccesary to design an additional grapheme, which demonstrated the empty slot or absence of a given magnitude order. That is, a symbol for *zero* had to be created.

Later in ([3], pág. 400) we find:

"The zero in a number, such as 1505, which as an empty column on the counting board had been familiar to the Greeks, the Romans and the medieval Western monks from earliest times, remained an obstacle which none of the three was able to overcome in a columnless system of numerals. The conceptual difficulty may have been this: The zero is something that must be there inorder to say that nothing is there".

Here Menninger reveals the other main limitation of the Roman numbering system: it did not inlcude a symbol for zero. Despite the consciousness of the quantity "absence", in leaving an empty column, the Roman system was unable to make the jump forward in defining a notation for zero, an issue that the Hindu-Arabic system resolved.

Sunya in Sanskrit, *as-sifr* in Arabic, *zefirum* in Latin, *chiffre* in French, *ziffer* in German, or *zero* in English. No matter the language we use to refer to the representation of the absence of quantity with a small circle, it was this idea that was behind the success of placed-valued systems, particularly Hindu-Arabic numbers. This is what Menninger calls *"encipherment"* throughout with his work.

Whether the idea of encipherment did or did not originate in India, truth is the idea of encipherment –or to locate between zeros– was first recorded in India with Brahmi and Gvailor numbers. The idea traveled to Persia to become Hindi/Gubar numbers thanks to the works of al-Kwārizmi, then it had the its frist migration attempt to Europe with Gerbert D'Aurillac, who introduced Apex Numbers. Finally, Leonardo Fibonacci succeeded at introducing Algorism numbers into a Europe that stubbornly refuseed to accept the advantages of the new enumeration of "unfaithful" devil worshippers.

Before moving on to the next section, where we describe the long and tedious journey that the Hindu-Arabic numbers made from India to Europe while passing through Arabia, we must mention that other civilizations had their own notation for zero, such as Babylonians and Mayas (See Menninger [3], pages 403 y 405).

Figure 2: The representation of zero by the Babylonic civilization (left) and Mayan civilization (right). Taken from Ifrah [2] pages 152 and 310.

4 The Migration to Europe

We have clarified that the Hindu-Arabic positional notation represented an epistemological jump from the Roman symbolic notation, concerning the individual assignment of a value to each figure of a unit, and the incorporation of zero as an additional grapheme that could represent the absence of quantity in the indicated magnitude order. In this section, we describe the journey of Hindu-Arabic notation and the main people that contributed to their spread. But first, it is necessary to reflect a little about the cultural resistance encountered during the assimilation process of the new numbers by the receiving culture. In Ifrah [2] we find in page 539 that:

"It is tempting to think that the Indian system spread through the Islamic world, replacing all other ways of representing numbers, and because of their ingenious simplicity, the corresponding calculation methods were rapidly accepted at all levels of Arab-Islamic society. The author humbly admits that he was wrong in the first edition of the present work in which he subscribed to the idea and neglected the following interesting details. Of course, certain scholars such as al-Kwārizmi and An Nisawi were suffiencitenly astute to understand the superiority of this system. But there was an equal number of Muslims who were, sometimes violently, opposed to the use of numerals and even more so to their becoming generalized".

Similarly, Medieval Europe, dominated by Catholic dogmatism, was not generous with the new numbers either. Menninger ([3], page 400) tells us that:

"Now let us follow the wanderings of the zero through space and time. There are few things to which the following quotation from Faust applies better than to this: Bei euch, ihr Herren, kann man das Wesen gewöhnlich aus dem Namen lesen, (In your case, my lords, the nature can generally be seen in the name). The zero, of course, is no Devil, but during the Middle Ages it was often regarded as the creation of the Devil".

As we can see, the assimilation in the first place of the Indian numbers by part of the Islamic world, and afterward of the Hindu-Arabic numbers by Medieval Europe, was nothing easy. On the contrary, due to cultural, religious prejudices, or fear of the unknown. The Hindu-Arabic notation finds a high resistance. In its long and complicated journey from India to Europe, with a previous pass through Arabia. It is incredible how the intolerance and fanatism of every type banned accepting a notation that possesses an evident advantage and fit, as we described above.

Among the main characters we owe the usage of the Hindu-Arabic notation, we find Abu Jaf'far Muhammad ibn Musa al-Khārizmi (circa 780-850), a Persian mathematician who wrote *al-Kitāb al-mukhasaṛ fī hịsāb al-jabr wal-muqābala*, where the word *algebra* comes. We know little from his life; he lived in the al-Ma'mum's court caliphate. And he was one of the most important mathematicians and astronomers that worked in the House of Wisdom (Ifrah [2], page 531). In this same page, Ifrah claims:

"His fame is due to two works which made significant contributions to the popularization of Indian numerals, calculation methods and algebra in both the Islamic world and the Christian West.".

We must remark in al-Kwhārizmi's times; the used species was *Gubar* numbers. The next illustration, taken from Ifrah's book, shows the current species of Eastern Hindu-Arabic numbers, known as the *Hindi numbers*. Currently, used in Libia, Egypt, Jordania, Siria, Saudi Arabia, Yemen, Lebanon, Iraq, Iran, among others (see Ifrah [2], page 368).

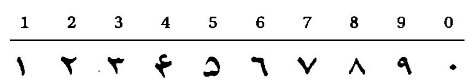
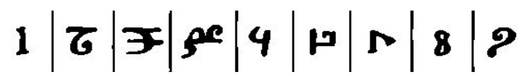


Figure 3: Extract from Ifrah's Book. Possibly, al-Khwārizmi used these numbers or a previous species.

In this work, we refer to this species as *Hindi-Gubar* númbers. Thanks to the works of al-Khwārizmi, the Indian numbers came to the Arabic culture, mainly represented by Persia and Arabia, from India. We call this the fist migration or first wave.

The second wave is from the Arabic-Muslim world in Persia and Arabia to general Catholic Europe. It is due to two main characters. Gerbert D'Aurillac (circa 946-1003) and Leonardo Fibonacci (1170-1250). Gerbert D'Aurillac, best known as the pope Sylvester II, was the first French pope of the Catholic church, he arrived at Saint Peter's throne in 999. As he was called the scientific pope, Gerbert was a pioneer in the introduction of Hindu-Arabic numbers thanks to his stay in the Santa María de Ripoll monastery near Barcelona during the Córdoba Caliphate. The next illustration, also taken from Ifrah's book, shows a sample of *Apex* species coming from several medieval sources (see Ifrah [2] page 580).

Figure 4: Extract from Ifrah's Book. XI century apex *Gerbertus, Raciones Numerorum Abaci.* Others apex have a notation for zero.



By his side, Leonardo Pisano, best known as Leonardo Fibonacci, was established with his father in Bugia, where he interacts with the positional notation and his comfort techniques calculation. He publishes in 1202 his famous *Liber Abaci (The Book of Abacus)*, being this work the primary historical reference until today in the Western World. The book deals with the introduction and arithmetic calculation with Hindu-Arabic numbers and their advantage against the Roman symbolic system.

The next illustration, coming from Ifrah, shows an example of the *Algorisms* species that predominated in Western Europe between the XII and XVII Centuries (see Ifrah [2] page 587). It is from this species we start to look at the similarities with the current species.

Figure 5: .Extract from Ifrah's Book. From the XII Century, *Algorism, Paris BN. Ms. lat 16202.* Almost all the species contain zero.

Undoubledly, the *Liber Abaci* mark the turning point for the beginning of acceptance of Hindu-Arabic notation in the low Middle Ages in Europe. It was only a matter of time before its ussage spread. It was primarily in the areas of traders where the positional notation settled. It makes sense because the simplification of transactions justified its use. By the end of the Middle Ages and the rise of the Renaissance, with the Scientific Revolution and the fantastic Age of Enlightenment, then ecclesiastic power had gradually decreased, then Hindu-Arabic notation finally founds the appropriate conditions to stabilize. It was a triumph of reason and common sense, but it was a long fight.

5 Evolution of the Notation

Next, we will take the illustrations from Ifrah's Book [2], in which the author displays the evolutionary track of the number species that led to the current notation. As with every evolution, there are species that extinct while others survived. In any case, the core idea that we want to present here as fiollows: First of all, the evolutionary line we introduced in the above sections starts with the *Brahami Numbers*, perhaps the first record of positional notation in India (See Menninger [3] page 397), where the *Gvalior Numbers* derived. Then, in Persia and Arabia, we find the *Hindi/Gubar Numbers*, which al-Khwārizmi possibly worked with. Later, we see the *Algorisms Numbers* used by Gerbert D'Aurillac until we reach the *Apex Numbers* during Fibonacci's era. The last species stemmed into the *Hindu-Arabic* numbers we know today. In summary, in this work, we have studied the sequence, proposed by Menninger [3].

Brahami Numbers ⟶ Gvalior Numbers ⟶ Hindi/ Gubar Numbers ⟶ Algorisms Numbers ⟶ Apex Numbers ⟶ Hindu-Arabic Numbers

Figure 6: Evolutionary history of the Hindu-Arabic numbers proposed by Menninger ([3], page 418).

Other evolutionary lines could have existed from Brahami numbers to Hindu-Arabic numbers. However, the second (and most important) item we want to emphasize is that the grapheme modificationof adding or subtracting the lines in the writing of numbers of the diverse species was what definitively led to evolutionary jumps among diverse species. We see in three the most representative case, which in Brahami notation are three horizontal bars. We believe that thoughout time, sciribes were gradually connecting the lines, until the representation of the number eventually evolved into its current representation. We do not want to speculate on how these graphemic alterations were incorporated into each evolutionary stage. The study of each figure's modifications is another topic of discussion, and it is not the goal of this work. We want to propose that the Hindu-Arabic notation we know was the result of cultural interaction and evolutionary history.

This seems reasonable, in contrast with the incoherent and nonsense explanations of Florian Cajori gave in *A History of Mathematical Notations* [1]. Many serious mathematicians, including Edouard Lucas (see [1] page 66), failed to explain each number's shape. Even today, these absurd explanations are seen throughout popular culture. A simple search on YouTube about the origin of numbers gives us a considerable quantity of explanations that only try to discuss the notation's current shape. But none allude to the vast number of species that existed throughout history. And, by one way or another, arrived in the form we know today. Not to mention mention, they completely ignore the cultural exchange over which the notation was exposed time: they try to explain *why* Hindu-Arabic numbers are the way they are, when it is *how* they evolved that is important and interesting.

Next, we show the individual evolutionary histories of each number from 1 to 9.

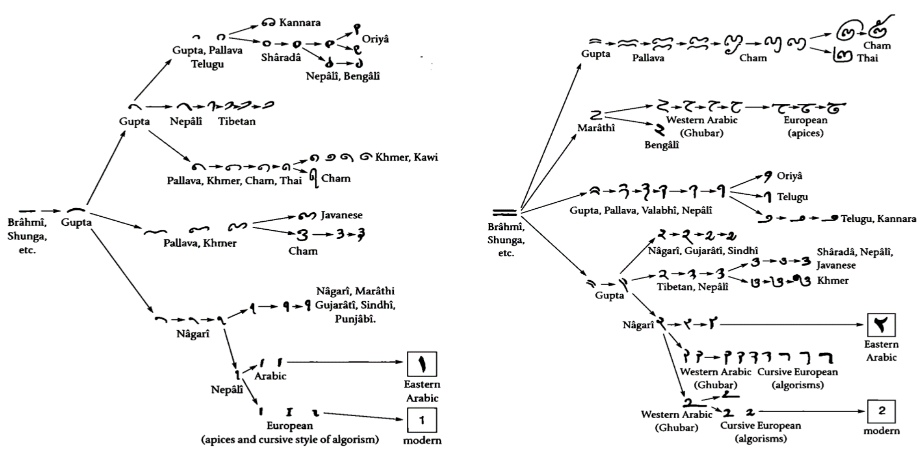


Figure 7: .Evolutionary history of 1 and 2 (See Ifrah[2], pages 392 and 393).

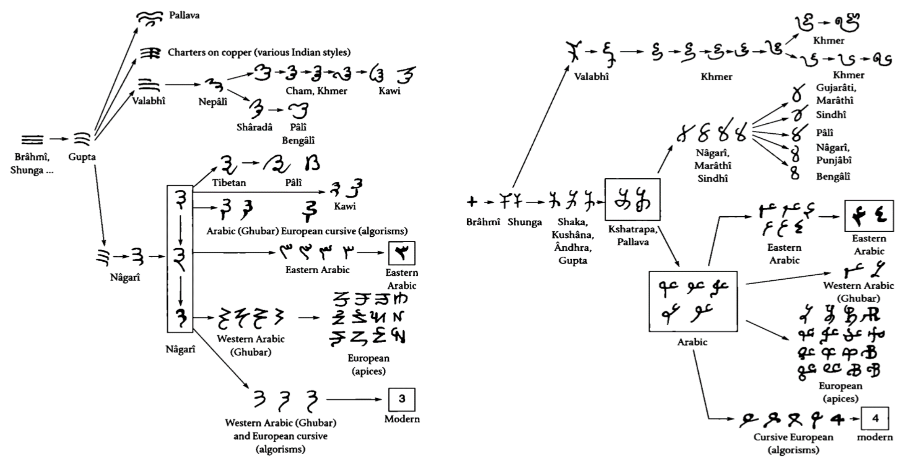


Figure 8: .Evolutionary history of 3 and 4 (See Ifrah[2], pages 393 and 394).

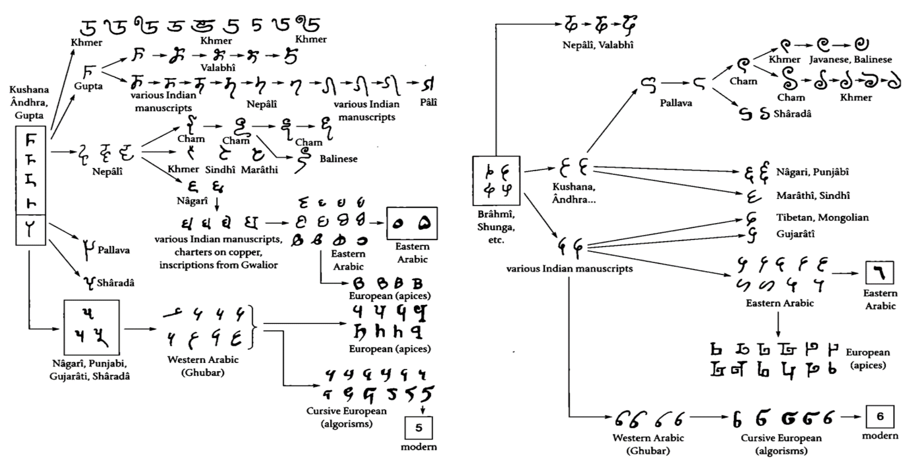


Figure 9: .Evolutionary history of 5 and 6 (See Ifrah[2], pages 394 ad 395).

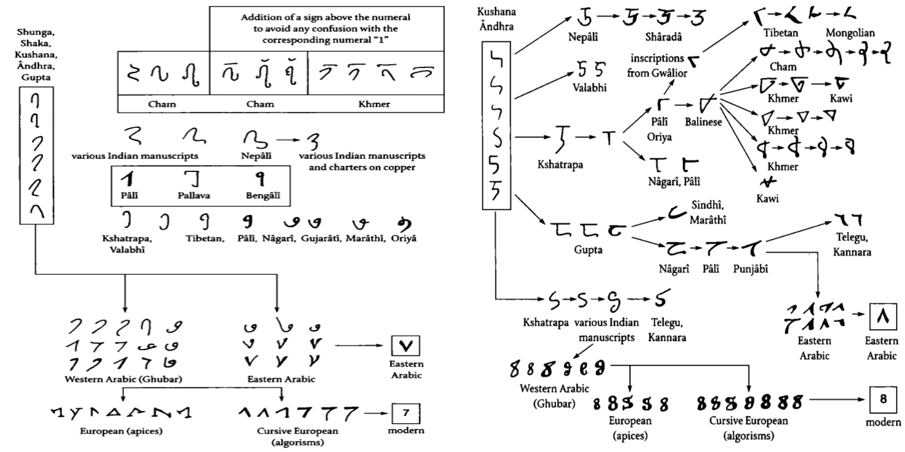


Figure 10: .Evolutionary history of 7 and 8 (See Ifrah[2], pages 395 and 396).

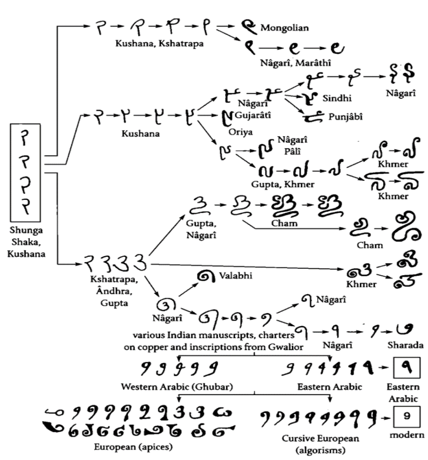


Figure 11: .Evolutionary history of 9 (See Ifrah[2], page 396).

In regards to zero's notation, in almost all species, it is an empty or full circle, as seen in some of the Apex species. Except Eastern Arabic Families, as we previously exposed in Hindi Numbers (See Figure 3), where it is represented with a dot. The notation of zero is not free from the world of speculations. In its series for BBC's *The Story of Maths*, the respected British mathematician Marcus Du Sautoy proposes that the symbol for zero comes from removing a buried stone in the sand. This hypothesis makes sense because if zero is the absence of quantity, to remove something from sand –represented by a stone, is equal to leaving it empty, and the stone-print in the sand is quite similar to zero. However, we neither support nor reject Du Sautoy's claim. Instead, we also propose that zero's representation had its own evolutionary story. Now, if we ask, Why the modern notation for zero is the one we know and not another that is another discussion.

6 Cultural Impact

Finally, we reach the point where we synthesize all exposed in the above sections. After defeating barriers, resistance to change, fanatism, fear of the unknown, and ignorance in general the Algorisms species finally evolved in the current species we know today. The next figure shows the *Modern* Hindu-Arabic Numbers.

0 1 2 3 4 5 6 7 8 9

Figure 12: The species of Modern Hindu-Arabic Numbers in Cambria Math font size 60.

There was no need to modify the notation in the last centuries. On the contrary, the computer era has improved the final design and the appearance of the modern species, practically without alteration. That gives to the modern species a universal category. In all places on earth use the modern specie is used to represent quantities, perhaps with the exception of Arabic Countries. They still use the Hindi/Gubar (See Figure 3). However, beyond the notation, Hindu-Arabic number's universality is the idea of a place-valued system that, after defeating all the barriers we already mentioned in the above sections, prevailed over the obsolete and unpractical symbolic Roman number system. The latter dominated the Western World from the rise of the Roman Empire to the Middle Ages in the XV Century. It is noteworthy to repeat that Hindu-Arabic numbers' predominance did not represent the extinction of Roman Numbers. Despite the decrease incurrent usage, they are used exclusively to represent ordinal numbers. The old Roman notation coexists without any issues with the modern Hindu-Arabic notation. Just common sense prevails in the human being, nobody in his or her right mind would use the nowadays the Roman numbers to perform arithmetical operations. The humans use the Hindu-Arabic notation for arithmetical calculations by its simplicity and comfort. To conclude, and from a very personal perspective, the author considers the Hindu-Arabic numbers as the most incredible legacy of the Indian civilization and Arabic civilization to the humankind.

7 Acknowledgments

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